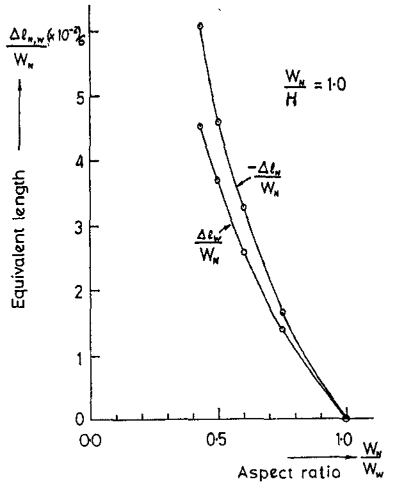

Fig. 3. Equivalent lengths of uniform lines ascribable to the impedance step.

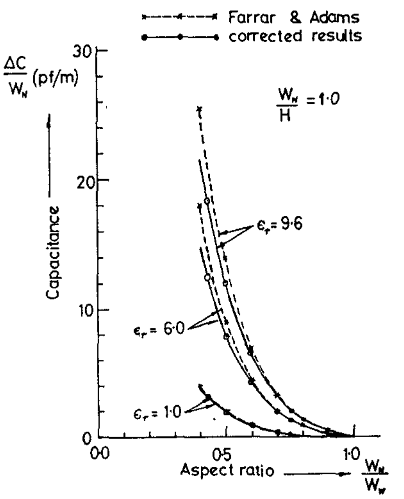

Fig. 4. Additional capacitance associated with the impedance step.

of lines of widths W_w and W_N, respectively, and C_{01} and C_{02} are the capacitances per unit length of the infinite (two-dimensional) lines of widths W_w and W_N. The two conjoining lines of lengths L create the region investigated, which has a total capacitance $C_t(L)$.

In light of lengths l_w and $-l_N$ calculated in the previous section, a corrected form of (6), accounting for the effects of inductance, is now

$$\Delta C = \lim_{L\to\infty} \left[C_t(L) - C_{oc}(1) - C_{oc}(2) - C_0(L + l_N) - C_{02}(L - l_N)\right]. \quad (7)$$

Results

The equivalent lengths l_w and $-l_N$, in normalized form, associated with a range of aspect ratios W_N/W_w, are shown in Fig. 3 for a constant ratio of $W_N/H = 1.0$, H being the strip-ground plane spacing, which was taken to be 0.025 in. These results supply the necessary information to establish a reference plane for the junction.

The results of Fig. 3 were then applied to (7), giving the corrections required of the capacitance values attained by Farrar and Adams, and these are given in Fig. 4, where their results can be seen to concur well for only a limited aspect ratio.

Of particular interest is the outcome of the comparison of computed results, in the case $\epsilon_r = 1.0$, for both (6) and (7), which indicates (within the numerical error limits) the dualities involved. This case of an absence of dielectric can be accounted for by the effect of series inductance or shunt susceptance on uniform transmission-line behavior, and one would suspect that the definition of an electrical reference plane for the junction would correspondingly produce the same answer. If, in the electrostatic case, the excess and deficiency of charge or capacitance on either side of reference plane T–T had been furnished, then for $\epsilon_r = 1.0$, lengths l_w and $-l_N$ could be discerned, and thus not only provide the reference plane required, but also make available self-correcting measures for the final calculation of capacitance, therefore producing a more representative equivalent.

In conclusion, extended information on the parameters associated with an impedance step have been presented, accounting for both inductance and capacitance, using static assumptions. The interpretation of dualities has enabled comparisons to be made with other theoretical work; these have proven to be extremely good. Furthermore, inferences may be drawn as to the complete adequacy of an exclusively electrostatic approach in the present application.

Acknowledgment

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Discontinuity Capacitance of a Coaxial Line Terminated in a Circular Waveguide: Part II—Lower Bound Solution

EDWARD W. RISLEY, JR.

***Abstract*—This calculation provides a lower bound (complementing the upper bound solution given earlier) to the discontinuity capacitance of a coaxial line terminated in a circular waveguide. A 50-Ω 0.9525-cm (3/4-in) open-circuited coaxial termination with a solid center conductor was fabricated with center- and outer-conductor diameters of 0.82723 ± 0.00005 and 1.90487 ± 0.00005 cm (1 cm = 0.393703 in), respectively. The measured value of capacitance of this termination at 1000 Hz was 216.4 ± 1.0 fF, as compared with the calculated lower bound of 215.0 fF. (The upper bound for this case was 217.7 fF.)**

I. Introduction

The standard of reflection for a coaxial line is the quarter-wave short-circuit termination. There are, however, shortcomings to this standard: the fabrication cost is high and each termination is usable at only one frequency. However, an open-circuited coaxial line with an extended outer conductor and a solid inner conductor could be used advantageously as a standard termination, because fabrication can be made using commercially available components and because the device is broad banded and losses are minimal. In addition to the high-frequency application, the device can also be used at low frequencies as a standard of capacitance.

In this short paper, the input impedance Z is formulated in terms of the magnetic field which leads to a lower bound solution for the discontinuity capacitance. Z is expressed as a stationary functional with a definite integral operator, and can therefore be shown to be bounded on the set of admissible trial functions [1].

This result complements the upper bound solution given in an earlier paper [2].

II. Integral Equation for Input Impedance

To derive the stationary form for the input impedance, assume an incident T wave propagating in the direction of increasing z (Fig. 1). Symmetry dictates that only E modes, independent of ϕ,

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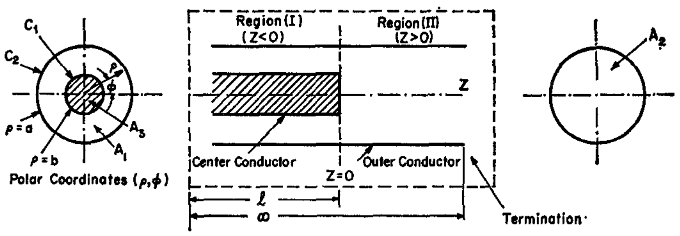

Fig. 1. Coaxial line terminated in a circular waveguide.

will be generated by the discontinuity at $z=0$. The modal expansions for the transverse components of the field in region I for $0 \leq \rho \leq a$ are

$$E_{\mathrm{I}} = a_0(e^{ikz} + \mathrm{R}e^{-ikz})\phi_0(\rho) + \sum_{n=1}^{\infty} a_n\phi_n(\rho)e^{-ih_nz} \tag{1}$$

$$H_{\mathrm{I}} = a_0(e^{ikz} - \mathrm{R}e^{-ikz})y_0\phi_0(\rho) - \sum_{n=1}^{\infty} a_ny_n\phi_n(\rho)e^{-ih_nz} \tag{2}$$

see [2] for a glossary of symbols, where the generalized Fourier coefficients are

$$a_0 = \frac{1}{y_0k^2(e^{ikz} - \mathrm{R}e^{-ikz})}\int_{A_2} H_{\mathrm{I}}\phi_0\, dA \tag{3}$$

$$a_j = \frac{e^{ih_jz}}{y_jh_j^2}\int_{A_2} H_{\mathrm{I}}\phi_j\, dA \tag{4}$$

with $\phi_0(\rho)$ and $\phi_n(\rho)$ defined to be zero on $[0, b]$.

Similarly, the transverse fields in region II for $0 \leq \rho \leq a$ are

$$E_{\mathrm{II}} = \sum_{n=1}^{\infty} b_n\psi_n(\rho)e^{ih_n'z} \tag{5}$$

$$H_{\mathrm{II}} = \sum_{n=1}^{\infty} b_ny_n'\psi_n(\rho)e^{ih_n'z} \tag{6}$$

where

$$b_j = -\frac{e^{-ih_j'z}}{y_j'h_j'^2}\int_{A_2} H_{\mathrm{II}}\psi_j\, dA \tag{7}$$

and it is assumed that all modes in this region are evanescent. To insure that this condition is satisfied, it is sufficient that the wavelength λ_{I} of the incident mode satisfies the inequality $\lambda_{\mathrm{I}} > 2\pi a/U_{\mathrm{I}}$, where U_{I} is the first zero of the equation $J_0(\gamma_1'a) = 0$.

Replace the coefficients in (1) and (5) by their integral expressions (3), (4), and (7); then

$$E_{\mathrm{I}} = \frac{(e^{ikz} + \mathrm{R}e^{-ikz})}{y_0k^2(e^{ikz} - \mathrm{R}e^{-ikz})}\phi_0(\rho)\int_{A_2} H_{\mathrm{I}}\phi_0\, dA + K_{\mathrm{I}}H_{\mathrm{I}} \tag{8}$$

$$E_{\mathrm{II}} = -K_{\mathrm{II}}H_{\mathrm{II}} \tag{9}$$

where K_{I} and K_{II} are the integral operators:

$$K_{\mathrm{I}}H_{\mathrm{I}} = \int_0^{2\pi}\int_0^a k_1(\rho, \rho')H_{\mathrm{I}}(\rho')\rho'\, d\rho' d\phi \tag{10}$$

$$K_{\mathrm{II}}H_{\mathrm{II}} = \int_0^{2\pi}\int_0^a k_2(\rho, \rho')H_{\mathrm{II}}(\rho')\rho'\, d\rho' d\phi \tag{11}$$

with kernels

$$k_1(\rho, \rho') = \sum_{n=1}^{\infty} \frac{\phi_n(\rho)\phi_n(\rho')}{y_nh_n^2} \tag{12}$$

$$k_2(\rho, \rho') = \sum_{n=1}^{\infty} \frac{\psi_n(\rho)\psi_n(\rho')}{y_n'h_n'^2} \tag{13}$$

which are separable, and therefore the integral transformations are completely continuous.

Continuity of transverse E across $z=0$ in A_1 requires

$$Z\phi_0(\rho)\int_{A_2} H_{\mathrm{I}}(\rho')\phi_0(\rho')\, dA + \int_{A_2} k_1(\rho, \rho')H_{\mathrm{I}}(\rho')\, dA = -\int_{A_2} k_2(\rho, \rho')H_{\mathrm{II}}(\rho')\, dA, \qquad \rho \in A_2 \tag{14}$$

where

$$Z = \frac{(1 + R)}{y_0k^2(1 - R)}.$$

Equation (14) contains implicitly the vanishing of tangential E on A_3.

Since $H_{\mathrm{I}} = H_{\mathrm{II}}$ on A_1, (14) can be rewritten as

$$Z\phi_0(\rho)\int_{A_2} \phi_0(\rho')H_{\mathrm{II}}(\rho')\, dA + \int_{A_2} k_1(\rho, \rho')H_{\mathrm{II}}(\rho')\, dA = -\int_{A_2} k_2(\rho, \rho')H_{\mathrm{II}}(\rho')\, dA$$

or

$$Z\phi_0(\rho)(\phi_0, H_{\mathrm{II}}) + kH_{\mathrm{II}} = 0 \tag{15}$$

where

$$KH_{\mathrm{II}} = \int_{A_2} k(\rho, \rho')H_{\mathrm{II}}(\rho')\, dA \tag{16}$$

and

$$k(\rho, \rho') = \sum_{n=1}^{\infty}\left\{\frac{\phi_n(\rho)\phi_n(\rho')}{y_nh_n^2} + \frac{\psi_n(\rho)\psi_n(\rho')}{y_n'h_n'^2}\right\}. \tag{17}$$

It follows from (15) that

$$Z(H, \phi_0)^2 + (H, KH) = 0$$

or

$$Z = -\frac{(H, KH)}{(H, \phi_0)^2} \tag{18}$$

where K is a definite symmetric operator, and hence (18) is a stationary form which is bounded.

III. Reduction of the Variational Expression

Since (18) is exact only if H is the actual field, expand H in the complete set $\{\psi_i(\rho)\}$:

$$H = \sum_{i=1}^{N} b_iy_i'\psi_i(\rho). \tag{19}$$

Equation (18) plus stationarity leads to the system

$$\sum_{i=1}^{N} b_iy_i'(Z\Omega_k\Omega_i + \Omega_{ki}) = 0, \qquad k = 1, 2, \cdots, N. \tag{20}$$

Equation (20) can be solved for Z to give

$$Z_k = -\frac{\Delta_k}{\sum_{i=1}^{k}\Omega_i\Delta_{ik}}, \qquad k = 1, 2, \cdots, N \tag{21}$$

where, for example, the determinants in (21) for $k=3$ are

$$\Delta_k = \begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{vmatrix}$$

$$\Delta_{1k} = \begin{vmatrix} \Omega_1 & \Omega_{12} & \Omega_{13} \\ \Omega_2 & \Omega_{22} & \Omega_{23} \\ \Omega_3 & \Omega_{32} & \Omega_{33} \end{vmatrix}$$

$$\Delta_{2k} = \begin{vmatrix} \Omega_{11} & \Omega_1 & \Omega_{13} \\ \Omega_{21} & \Omega_2 & \Omega_{23} \\ \Omega_{31} & \Omega_3 & \Omega_{33} \end{vmatrix}$$

$$\Delta_{3k} = \begin{vmatrix} \Omega_{11} & \Omega_{12} & \Omega_1 \\ \Omega_{21} & \Omega_{22} & \Omega_2 \\ \Omega_{31} & \Omega_{32} & \Omega_3 \end{vmatrix}$$

The determinant elements are defined in terms of the inner products

$$\Omega_i = (\phi_0, \psi_i) \tag{22}$$

$$\Omega_{ij} = \beta\sum_{n=1}^{N}\left\{\frac{(\psi_i, \phi_n)(\phi_n, \psi_j)}{\alpha_n} + \delta_{in_j}\alpha_n'^3\right\} \tag{23}$$

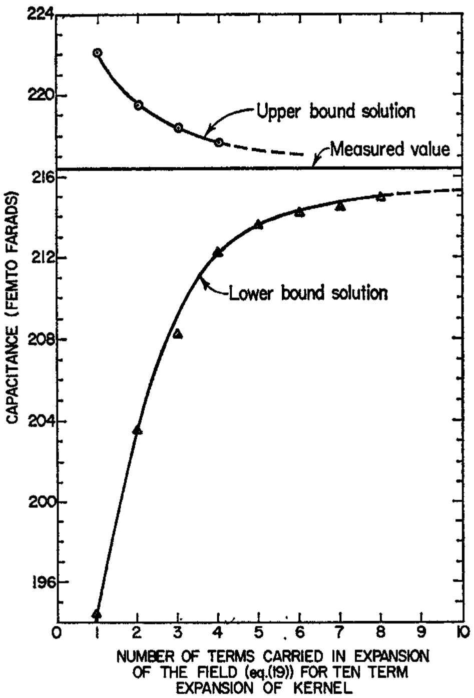

Fig. 2. Convergence behavior of upper and lower bound solutions.

where

$$\delta_{inj} = \begin{cases} 1, & i = n = j \\ 0, & \text{otherwise.} \end{cases}$$

IV. Results

Equation (21) was evaluated using an electronic computer. A 50-Ω 0.9525-cm (3/4-in) open-circuited coaxial termination with a solid center conductor was fabricated with center- and outer-conductor diameters of 0.82723 ± 0.00005 and 1.90487 ± 0.00005 cm (1 cm = 0.393701 in), respectively. The measured value of capacitance of this termination at 1000 Hz was 216.4 ± 1.0 fF, as compared with the calculated lower bound of 215.0 fF. [The upper bound for this case was 217.7 fF (see 2).] The number of terms carried in the expansions for H and $k(\rho, \rho')$ were eight and ten, respectively.

Fig. 2 is a plot of the calculated value of capacitance as a function of the number of terms carried in the expansion of the field [see (19)] for a ten-term expansion of the kernel [see (17)]. Also displayed is the convergence behavior of the upper bound solution.

The error bounds provided by this method make it useable for standards work. In other methods, error bounds must be inferred from the convergence behavior of the solution. The minimum error bounds determinable by this method must wait until funds become available. In theory, of course, this limit could be reduced to zero.

Somlo [3] obtained a value of 216.8 fF using a 40-term expansion of the series derived by Whinnery *et al.* [4]. This value lies between the upper and lower bounds obtained here.

V. Nomenclature

A_1, A_2, A_3 See Fig. 1.
E Radial component of transverse electric field.
H Transverse component of magnetic field.
R Reflection coefficient.
a_0 Amplitude of the incident wave in region I.
h_n $=\sqrt{k^2-\gamma_n^2}=i\alpha_n$; $\alpha_n=\sqrt{\gamma_n^2-k^2}$.
h_n' $=\sqrt{k^2-\gamma_n'^2}=i\alpha_n'$; $\alpha_n'=\sqrt{\gamma_n'^2-k^2}$.
k $=\omega\sqrt{\mu\epsilon}$.
y_n $=\omega\epsilon/h_n$—wave admittance corresponding to the nth mode in the region $Z<0$.
y_n' $=\omega\epsilon/h_n'$—wave admittance corresponding to the nth mode in the region $Z>0$.
y_0 Characteristic admittance.
$\psi_n(\rho)$ Mode function of the nth mode in a circular waveguide.
γ_n nth eigenvalue corresponding to the eigenfunction $\Phi_n(\rho)$ (see I).
γ_n' nth eigenvalue corresponding to the eigenfunction Ψ_n (see I).
ϵ Dielectric constant.
ρ Polar coordinate.
μ Permeability.
φ Polar coordinate.
$\varphi_0(\rho)$ Mode function of the dominant mode in a coaxial line.
$\phi_n(\rho)$ Mode function of the nth mode in a coaxial line.
ω $=2\pi f$; f = frequency.

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On Inhomogeneously Filled Rectangular Waveguides

K. F. CASEY

***Abstract*—A method is given for determining the characteristic equations and field components of the LSE and LSM modes in rectangular waveguides filled with a dielectric which is inhomogeneous in one transverse dimension. The method is exact and yields solutions for a nearly arbitrary variation in permittivity across the waveguide.**

Propagation in waveguides which are inhomogeneously filled in the transverse direction has been of interest for many years, because of applications to a variety of microwave components, including phase changers, matching transformers, and quarter-wave plates [1]. In these applications, the inhomogeneous loading is generally accomplished by partially filling the guide cross section with a dielectric slab. There has also been some attention given to the more general problem in which the permittivity variation is continuous over one dimension of the guide cross section [2], [3]. In this short paper, we consider this more general situation and present a method by which the electromagnetic fields may be determined for a nearly arbitrary variation of permittivity across the waveguide.

Consider a rectangular waveguide formed by conducting surfaces at $x=0$ and $x=a$ and $y=0$ and $y=b$. The material filling the guide is an inhomogeneous dielectric of permittivity $\epsilon(x)$ and permeability μ_0. Assuming a time dependence exp $(j\omega t)$, the LSE modes are obtained from

$$\bar{E} = \nabla \times \Phi \bar{a}_x \tag{1a}$$

$$\bar{H} = \frac{1}{j\omega\mu_0} \nabla \times \nabla \times \Phi \bar{a}_x \tag{1b}$$

where

$$\nabla^2 \Phi + k^2(x)\Phi = 0 \tag{2}$$

with $k^2(x) = \omega^2\mu_0\epsilon(x)$. The elementary product solutions of (2) are given by

$$\Phi(x, y, z) = f(x) \cos \frac{n\pi y}{b} e^{-j\beta z} \tag{3}$$

in which $n = 0, 1, 2, \cdots$, β is the propagation constant in the axial

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